FEATURES OF THE ELECTRIC CURRENT AND FIELD DISTRIBUTIONS IN FLOWS WITH NARROW LAYERS OF HIGHLY CONDUCTIVE GAS

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A solution obtained by Fourier's method provides the basis for analyzing the influence of a narrow gas layer, of higher conductivity than the rest of the flow, on the Joule dissipation and current distribution in the terminal zone of a plane magnetohydrodynamic channel with nonconducting walls. The MHD interaction parameter, Reynolds magnetic number, and Hall parameter are assumed small. It is shown that a narrow, highly conductive layer can on occasions be replaced by a surface of discontinuity, on which well-defined relations between the electric quantities are satisfied. The presence of such a layer leads to an increase in the Joule dissipation and a reduction in the lengths of the current lines. A hopeful arrangement for a magnetohydrodynamic energy converter is one in which an inhomogeneous flow is used, consisting of a continuous series of alternating very hot and less hot zones [1,2]. For this arrangement, it is worth examining the influence of the stratified conductivity distribution of the working body on the Joule dissipation and the electric currents in the channel. Numerous papers have discussed the case of inhomogeneous conductivity in the context of MHD system electrical characteristics. A general solution was obtained in [3] for the stationary problem on the electric field in a plane MHD channel with nonconducting walls when the magnetic field and conductivity are arbitrary functions of the longitudinal coordinate. In [4], where the braking of undeformed conducting clusters was investigated, the Joule dissipation, linked with the appearance of closed eddy currents in the cluster as it enters and leaves the magnetic field, was evaluated. The relationships between the electrical quantities, on moving through a narrow layer of low-conductivity liquid, were considered in [5].

1. Consider a plane channel $-\infty < x < +\infty$, $0 \le y \le h$ with nonconducting walls (Fig. 1), through which flows a medium with stratified electrical conductivity: σ_1 in regions 1 and 3 (the less hot zones), and σ_2 in region 2 (the hotter zone). The flow takes place in a transverse magnetic field $B = (0,0,B_z(x))$ with weak magnetohydrodynamic interaction. In this case the hydrodynamic quantities can be assumed known, and only the equations of electrodynamics have to be solved. For simplicity, let the movement take place with constant velocity $V = V_x = \text{const}$. In addition, let the Reynolds magnetic number R_m and the Hall parameter $\omega\tau$ be small. Under these assumptions, the electric current j and potential φ distributions are described by the equations

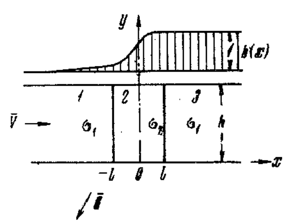

Fig. 1

$$\mathbf{j} = \sigma\,(-\nabla\varphi + c^{-1}\mathbf{V}\times\mathbf{B}),\qquad \operatorname{div}\mathbf{j} = 0\,, \tag{1.1}$$

From (1.1), the Laplace equation

$$\Delta\varphi = 0 \tag{1.2}$$

holds for φ.

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As the boundary conditions on the walls we have the conditions for no-flow of the normal current

$$j_y = 0, \qquad \frac{\partial \varphi}{\partial y} = -\frac{1}{c} V_x B_z(x) \quad \text{when } y = 0, h \; . \tag{1.3}$$

We introduce dimensionless variables defined by

$$\begin{gathered} x = x^\circ h, \quad y = y^\circ h, \quad l = l^\circ h, \quad V = vV^* \\ v = 1, \quad B = bB^*, \sigma = \sigma^\circ \sigma_1, \quad \varepsilon = \sigma_1/\sigma_2 \\ \varphi = \varphi^\circ \frac{V^* B^* h}{c}, \quad j = j^\circ \frac{\sigma_1 V^* B^*}{c}, \quad Q = Q^\circ \frac{\sigma_1 V^{*2} B^{*2} h^2}{c^2} \; . \end{gathered} \tag{1.4}$$

Here V* and B* are the characteristic velocity and induction of the magnetic field, and Q the Joule dissipation. For convenience, the degree sign will be omitted in the dimensionless variables. We introduce the auxiliary potential

$$\psi = \varphi + b(x) y \; . \tag{1.5}$$

The boundary condition (1.3) on the walls becomes homogeneous when written for ψ. The problem may easily be solved by Fourier's method; the required function $\psi(x,y)$ is written as a series in each of the regions 1, 2, and 3:

$$\begin{gathered} \psi_i(x, y) = \sum_{\nu=1}^{\infty} \psi_{\nu i}(x) \cos(2 r_\nu y) + \frac{b(x)}{2} \qquad (i = 1, 2, 3) \\ r_\nu = {}^1\!/_2 \pi (2\nu - 1) \qquad (\nu = 1, 2, \ldots, n, \ldots) \; . \end{gathered} \tag{1.6}$$

On substituting (1.5) and (1.6) in (1.2), ordinary second-order differential equations are obtained for the functions $\psi_{\nu i}(x)$, which may be solved under the boundary conditions resulting from the boundaries of the potential at $\pm\infty$ and the discontinuity of the normal current j_x and of the tangential component $E_y = \partial\varphi/\partial y$ on the lines of discontinuity $x = -l$ and $x = +l$.

The solution can then be used to analyze the behavior of the electrical quantities on passing through the strip $(-l, l)$ when the width of the strip is small ($l \to 0$), the conductivity σ_2 is high ($\varepsilon \to 0$), and we have different ratios l/ε.

2. Consider an elementary volume in the strip $(-l, l)$, of unit width, length $2l$, and height dy. The integral equation

$$\int_v \operatorname{div} \mathbf{j} \, dv = 0$$

holds for this volume, or by the Gauss-Ostrogradskii theorem,

$$\int_\Sigma \mathbf{j} \, ds = 0 \; . \tag{2.1}$$

The relationship

$$j_x(-l, y) - j_x(+l, y) = \frac{d}{dy} \int_{-l}^{+l} j_y(x, y) \, dx = \frac{dI}{dy} \tag{2.2}$$

can be obtained from (2.1), where I is the current through the strip cross section. Substituting the expression for j_y obtained from (1.1) in (2.2), we get

$$\frac{dI}{dy} = -\frac{1}{\varepsilon} \int_{-l}^{+l} \frac{\partial^2 \varphi}{\partial y^2} \, dx \; . \tag{2.3}$$

If b(x), representing the magnetic field as a function of the longitudinal coordinate, is continuous everywhere and bounded along with its derivatives b'(x) and b''(x), we find by passing to the limit as $l \to 0$ in the solution obtained in Sec. 1, that

$$j_y(0, y) = \frac{1}{2\varepsilon} \sum_{\nu=1}^{\infty} \frac{K \sin(2r_\nu y)}{r_\nu^2 (1 + 2r_\nu l/\varepsilon)}, \qquad \frac{\partial\varphi}{\partial y}(0, y) = -\varepsilon j_y(0, y) - b(0) \tag{2.4}$$

$$j_x(-l, y) - j_x(l, y) = \frac{2l}{\varepsilon} \sum_{\nu=1}^{\infty} \frac{K \cos(2r_\nu y)}{r_\nu (1 + 2r_\nu l/\varepsilon)}, \qquad I = 2l j_y(0, y) \tag{2.5}$$

$$K = 4r_\nu \left[r_\nu \left(\int_{-\infty}^{0} b(x) \exp(2r_\nu x)\, dx + \int_{0}^{\infty} b(x) \exp(-2r_\nu x)\, dx \right) - b(0) \right]. \tag{2.6}$$

It also follows from the passage to the limit that the difference $\partial\varphi/\partial y(l, y) - \partial\varphi/\partial y(-l, y) \sim l$ and tends to zero as $l \to 0$. The same conclusion can be reached for the difference between the potential φ values at the strip edges, since the problem is symmetric about the line $y = 1/2$.

Comparing (2.4) and (2.5), the relationship

$$\frac{\partial\varphi}{\partial x}(l, y) - \frac{\partial\varphi}{\partial x}(-l, y) = -\frac{2l}{\varepsilon} \frac{\partial^2\varphi}{\partial y^2}(0, y) \tag{2.7}$$

is obtained.

Notice that, according to the second of Eqs. (2.5), the current density $j_y(x,y)$ varies weakly along the strip. This means that (2.7) may be obtained directly from (2.2) and (2.3), by replacing the integrals in them by the integrands, taken at the mid-point of the strip, multiplied by the strip width. In cases when $j_y(x,y)$ varies substantially along the strip (e.g., when the magnetic field has a discontinuity inside the strip), this substitution cannot be made, and the relationship (2.7) may not be satisfied. A similar condition was obtained earlier by Shercliff [6] when analyzing the boundary conditions on thin, highly conductive, fixed walls, contacting a conducting liquid on one side and a fixed nonconducting space on the other. The influence of the contact impedance between the liquid and wall was examined. Since (2.7) has the same structure as Shercliff's condition, we shall refer to it by that name.

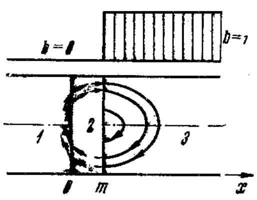

Fig. 2

To sum up, in the presence of a narrow strip possessing very high conductivity, one of the usual relationships at a discontinuity, namely, continuity of the tangential component of the electric field, always remains valid, while the second relationship, namely, continuity of the current density component normal to the discontinuity, is violated when the ratio l/ε is finite or tends to infinity. This result should cause no surprise in the present case, since the condition $\{j_n\} = 0$ is traditionally derived on the assumption that j_τ is bounded on the surface of discontinuity, whereas in our example j_y increases indefinitely at interior points of the strip.

Letting $l/\varepsilon \to \infty$, (2.4) gives

$$\frac{\partial\varphi}{\partial y}(0, y) + b(0) = 0\,. \tag{2.8}$$

Since the coordinate x = 0 on the center-line of the strip, (2.8) implies that

$$\varphi = \text{const} \tag{2.9}$$

on the strip, regardless of the magnetic field.

This boundary condition is usually specified on electrodes.

In short, when solving magnetohydrodynamic problems concerned with flows with narrow layers of highly conductive medium, the following procedure may be adopted; if $l/\varepsilon \to 0$, the layer can be disregarded and the working performed in the usual way; while if l/ε is finite or tends to infinity, the layer may be replaced by a surface of discontinuity, on which the conditions $\{\partial\varphi/\partial y\} = 0$ and one of conditions (2.7), (2.8), or (2.9), are satisfied, depending on the size of the ratio l/ε andthe value of b(0). It is clear from what has been said that these recommendations should only be followed when the functions b(x), b'(x), and b"(x) are continuous and bounded at interior points of the highly conductive layer (or layers).

Notice the following point. It was mentioned in Sec. 1 that the problem was being solved on the assumption that the induced magnetic fields could be ignored. But the conductivity σ_2 of the strip $(-l,l)$ is assumed to be quite high, so that $R_m(\sigma_2)$ may in general be considerably greater than unity. It was therefore necessary to evaluate the induced magnetic field B_i that may be produced by currents flowing in the strip. Direct evaluation of the induced magnetic field from the computed currents showed that, regardless of the strip conductivity σ_2, we always have $B_i \ll B$, so that it may in fact be ignored in the calculations. This is bound up with the following features of the current flow in the end zones of the magnetic field. The current I flowing in the strip depends on the position of the latter relative to the applied field. If the strip is at the center of the end currents (K = 0), no current flows in it. If the strip is left of center (K > 0) or right of center (K < 0), the current flowing in the highly conductive strip will also flow in the adjacent low-conductivity regions, and the total impedance to the current path proves to be quite high.

3. Consider the influence on the Joule dissipation Q when a strip of high conductivity is situated in a medium of low conductivity. In a channel with nonconducting walls, Q is well known to be [3]

$$Q = -2a \int_0^1 \int_{-\infty}^{+\infty} j_y b\,dx\,dy \tag{3.1}$$

where $2a$ is the channel width.

Substituting (1.1), (1.5), and (1.6) in (3.1), we get

$$Q = -4a\sigma \sum_{\nu=1}^{\infty} \int_{-\infty}^{+\infty} \psi_\nu(x)\, b(x)\, dx\,. \tag{3.2}$$

It has been shown in several earlier papers that, the more sharply the magnetic field drops in the entry and departure zones, the greater the end currents flowing in the channel. It therefore seems worth considering the model problem in which the magnetic field has a step-wise dependence on the longitudinal coordinate. Let the strip $(-l,l)$be located at a distance m from the point where the magnetic field jumps (Fig. 2). Divide the channel into three regions, as shown in Fig. 2. As before, the solution for the function $\psi(x,y)$ in each of the regions will be sought as a series (1.6). The ordinary second-order differential equations obtained during the process of solution will be solved by utilizing the conditions on the discontinuities and the boundedness conditions for the potential at $\pm\infty$. The usual relationships at a discontinuity will be employed on the line x = m, while the conditions $\{\partial\varphi/\partial y\} = 0$ and Shercliff's condition (2.7) will be used on the line x = 0. The result is the following for the Joule dissipation:

$$Q = 2a \sum_{\nu=1}^{\infty} \frac{1}{r_\nu^3} - a \sum_{\nu=1}^{\infty} \frac{\operatorname{sh}(2r_\nu m)(1 + 4r_\nu l/\varepsilon) + \operatorname{ch}(2r_\nu m)}{r_\nu^3 (1 + 2r_\nu l/\varepsilon)[\operatorname{sh}(2r_\nu m) + \operatorname{ch}(2r_\nu m)]}\,. \tag{3.3}$$

Consider some limiting cases.

<u>First limiting case:</u>

$$\frac{l}{\varepsilon} \to 0, \qquad Q = a \sum_{\nu=1}^{\infty} \frac{1}{r_\nu^3}\,.$$

The solution is the same as for the corresponding problem in which no strip of high conductivity is present.

Second limiting case:

$$\frac{\varepsilon}{l} \to 0, \qquad Q = 2a \sum_{\nu=1}^{\infty} \frac{1}{r_\nu^3 [1 + \operatorname{th}(2r_\nu m)]} \, .$$

In this case Q has a maximum as $m \to 0$, equal to

$$Q = 2a \sum_{\nu=1}^{\infty} \frac{1}{r_\nu^3} \, .$$

Note that the passage to the limit $m \to 0$ implies that the strip $(-l, l)$ comes infinitesimally close to the point where the magnetic field is discontinuous, though the point of discontinuity remains outside the strip.

Third limiting case:

$$m \to \infty, \qquad Q = a \sum_{\nu=1}^{\infty} \frac{1}{r_\nu^3} \, .$$

The highly conductive layer is seen to lead to an increase in the Joule dissipation; the closer it is to the point where the magnetic field is discontinuous, the greater its influence. As $m \to \infty$, the layer has no effect on Q, since the end currents are substantially nonzero at distances of the gage order (h) from the point where the magnetic field changes rapidly.

4. Consider the influence of a narrow layer of high conductivity on the current distribution in the channel. Suppose, for instance, that the layer $(-l, l)$ is to left of center of the end currents, where $K > 0$, and that a current I flows through it in the positive direction of the y axis. Noting that $\cos(2r_\nu y)$ is an odd function relative to the line $y = 1/2$, we obtain from (2.5)

$$\begin{aligned} j_x(-l, y) - j_x(l, y) > 0 \quad \text{when} \quad 0 < y < 1/2 \\ j_x(-l, y) - j_x(l, y) < 0 \quad \text{when} \quad 1/2 < y < 1 \, . \end{aligned} \tag{3.4}$$

Since $j_x \leq 0$ when $0 < y < 1/2$, $|x| < \infty$, and $j_x \geq 0$ when $1/2 < y < 1$, $|x| < \infty$, it follows from the inequalities (3.4) and the continuity of $\partial\varphi/\partial y$ and b on passing through the narrow layer, that the current lines will be deformed in the manner shown in Fig. 2. It is clear from Fig. 2 that the conducting layer leads to a decrease in the length of a current line, since the latter is, so to speak, drawn in towards the layer.

The solutions of the problems considered in Secs. 1 and 3 show that, as $\varepsilon/l \to 0$, the end currents are closed up through the layer, i.e., no currents flow in the region 1. When $l/\varepsilon \to 0$, the picture of the current lines is the same as when no layer of high conductivity is present. The usual picture of current flow in the channel is likewise retained when the layer is remote from the region in which the magnetic field changes.

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